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The momentum flux in turbulent submerged jets

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Although the jet momentum flux has been traditionally accepted as constant, this is not in general true because a weak pressure field is induced in the ambient fluid with positive gradient and because the induced flow field carries momentum flux to the jet. The angle ϕ, at which the induced flow streamlines enter the jet, is the basic parameter which determines whether the jet momentum flux increases, remains constant or decreases. A theoretical solution is presented for the variation of the jet momentum flux in turbulent submerged jets in stationary ambient fluid. The solution presented in this paper generalizes previous theoretical solutions and is in good agreement with existing experimental results. The contribution of the induced pressure field relative to the induced velocity field in varying the jet momentum flux is investigated. The induced flow streamlines are calculated using non-constant jet momentum flux and are compared with Taylor's solution (where constant jet momentum flux was assumed).

1. Introduction

For many years it was believed that the momentum flux in any jet is very nearly constant (e.g. Townsend 1976; Schlichting 1960; Rajaratnam 1976). Although the theoretical derivation of this statement is based on a number of assumptions (which are discussed below), many investigators viewed the constancy of momentum as an exact, unquestionable statement and used it to calibrate their Pitot tubes (see Flora & Goldschmidt 1969), or as criterion for the acceptance of 'reliable' data (see Rodi 1975, p. 105). The theoretical hypotheses that lead to the constancy of the jet momentum flux are basically two: first, ambient pressure p is everywhere hydrostatic and, second, the induced flow field always has streamlines perpendicular to the jet axis. The first hypothesis is never true because there must be a small but significant pressure difference to establish the movement of surrounding fluid into the jet. The second hypothesis is not in general true. These two approximations simplify the theoretical examination of the turbulent jet but introduce an error that has to be determined.

To the best of our knowledge the assumption of the constancy of the jet momentum flux was questioned for the first time by Kotsovinos (1975) who argued that there are two factors which act on the jet momentum flux: (i) the pressure field generated in the ambient fluid, which always decreases the jet momentum flux; (ii) the induced flow towards the jet which increases or decreases the jet momentum flux, depending on the angle ϕ with which the induced flow streamlines enter the jet.

Kotsovinos (1975, p. 37) estimated the decrease of the momentum flux in a plane jet assuming that the entrainment into the jet (i.e. the jet volume flux) depends on the decreased local momentum flux (i.e. considering the effect which the decrease of the momentum flux exerts on the induced flow). However, he simplified the problem by assuming that the induced pressure field in the ambient fluid is negligible. His solution for a plane jet is the same as the solution reported later by Schneider (1985), who arrived at his results using a different approach to the problem (inner and outer expansions). We will show in this paper that it is not a good approximation to neglect the induced pressure. Kotsovinos (1978*b*) included the contribution of the induced pressure in decreasing the jet momentum flux. However, he simplified the mathematical solution by assuming that the entrainment into the jet (i.e. the jet volume flux) depends on the initial momentum flux. He applied these ideas to the plane jet out of a wall and he found that to a first approximation

$$M(x)/M_e = 0.983-0.0693\ln(x/6D), \tag{1}$$

where $M(x)$ and M_e are the momentum fluxes at distances x and zero respectively from the jet exit, and D the slot thickness of the plane jet. The momentum flux given by (1) vanishes at a distance $x/D = 10^8$ (i.e. a distance which is very large for practical applications). However, mathematically (1) is not precise because, as Schneider (1985) pointed out, the solution given by it breaks down as the distance from the jet orifice tends to infinity, i.e. $M(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

Therefore it appears that there are open questions regarding the theoretical investigation of the variation (increase or decrease) of the jet momentum flux. The purpose of this paper is to present a precise solution for the variation of the jet momentum flux $M(x)$ with the axial distance x, and to improve our understanding of the important role of the induced ambient pressure. The 'coupling' of the jet momentum flux with the induced flow streamlines is also explored and a solution is presented for the induced flow field based on non-constant jet momentum flux.

2. Analysis of the problem

Without loss of generality we restrict our analysis to two-dimensional (plane) jets. Basic results are given by Kotsovinos (1978*b*). A plane jet is defined as a source of kinematic fluxes of mass V_e and momentum M_e (per unit span) through a slot of thickness D into a finite space filled with fluid of density ρ. It is assumed that the jet is turbulent and that the ambient fluid is quiescent, except for flows induced by the presence of the jet. The system of coordinates and the various symbols are defined in figure 1. The jet boundaries can be described by the equation $y = \pm B(x) = \pm kx$ (see figure 1), where k is an experimental constant, approximately equal to 0.25. Mean flow and turbulent quantities are denoted by a bar and by a prime, respectively. The jet volume flux $V(x)$ is defined as

$$V(x) = \int_{-B(x)}^{B(x)} \bar{u}(x,y)\,\mathrm{d}y. \tag{2}$$

The induced flow streamlines enter the jet at an angle ϕ (see figure 1 for definition). The axial momentum equation acros the jet can be integrated from $x = 0$ to x to obtain (for a detailed proof see Kotsovinos 1978*b*)

$$M(x) = M_e + C(x) + H(x), \tag{3}$$

where the term

$$M(x) = \int_{-B(x)}^{B(x)} \left(\bar{u}^2 + \overline{u'^2} + \frac{p}{\rho}\right)\mathrm{d}y \tag{4}$$

is the kinematic momentum flux of the jet (or as Benjamin 1968 named it, the flow force of the jet); M_e is the input momentum flux; the term

$$C(x) = \int_0^x 2(-\bar{u}\bar{v} + k\bar{u}^2)\,\mathrm{d}x = \int_0^x \frac{\tan\phi}{2(1-k\tan\phi)}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{5}$$

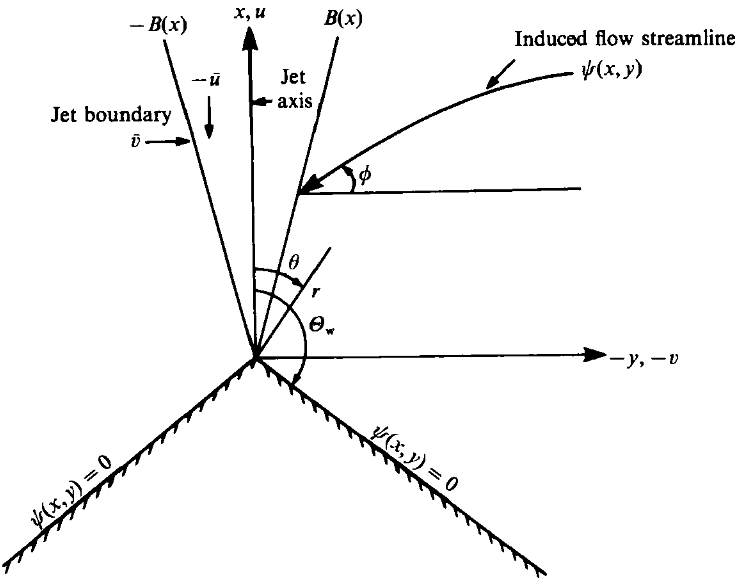

FIGURE 1. Geometry of the plane jet.

is the contribution of the induced flow momentum flux, calculated at the jet boundaries, which can be either positive or negative (i.e. the momentum of the induced flow field increases or decreases the jet momentum flux $M(x)$) depending on the induced flow angle ϕ; and the term

$$H(x) = \int_0^x 2k\left(\frac{p}{\rho}\right)\mathrm{d}x = -\int_0^x \frac{k(1+\tan^2\phi)}{4(-1+k\tan\phi)^2}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{6}$$

is the contribution of the mean pressure $p(x, kx)$ generated in the ambient fluid and calculated at the jet boundaries. This term is always negative (i.e. always tends to decrease the jet momentum flux).

Therefore, the jet momentum flux $M(x)$ may increase, remain constant, or decrease, according to whether the sum $C(x)+H(x)$ is respectively larger, equal to or smaller than zero.

Combining (5) and (6), the jet momentum flux (3) can be written:

$$M(x) = M_e - \int_0^x \left[\frac{k(1+\tan^2\phi)}{4(-1+k\tan\phi)^2} - \frac{\tan\phi}{2(-1+k\tan\phi)}\right]\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 \mathrm{d}x. \tag{7}$$

The problem of integrating (7) is therefore reduced to finding an appropriate expression for the induced flow angle $\phi(x)$ and for the increase of volume flux $\mathrm{d}V/\mathrm{d}x$.

Taylor (1958) examined the flow induced by a fully developed jet in an infinite medium. He found that the induced flow streamlines represent parabolas which enter the jet axis at a constant angle ϕ. In what follows we shall adopt this finding of constant induced flow angle ϕ, and discuss it in more detail in §3. For $0 < x < x_0$, where $x_0 = 6D$, the jet flow has not fully developed and the increase in volume flux $\mathrm{d}V/\mathrm{d}x$ is given by (see Liepmann & Laufer 1947; Hussain & Clark 1977)

$$\frac{\mathrm{d}V}{\mathrm{d}x} = C_1\left(\frac{M_e}{D}\right)^{\frac{1}{2}},$$

where C_1 is an experimental constant which strongly depends on the initial boundary layers and varies between 0.06 to 0.16.

Integrating (7) from $x = 0$ to $x = x_0$ we find the jet momentum flux at the end of the flow development region:

$$M_0 = M(x_0) = M_e - (C_1)^2 \left(\frac{x_0}{D}\right) \left[\frac{k(1+\tan^2\phi)}{4(-1+k\tan\phi)^2} - \frac{\tan\phi}{2(-1+k\tan\phi)}\right] M_e. \tag{8}$$

For $C_1 = 0.07$, $x_0 = 6D$ and for $\phi = 45°$, we find $M_0 = 0.974 M_e$ and for $\phi = 0°$, $M_0 = 0.988 M_e$.

For $x > x_0$, the jet flow is effectively fully developed and the increase in volume flux dV/dx, due to the entrainment of ambient fluid, is given (on dimensional grounds) by the equation

$$\frac{dV}{dx} = \epsilon^{\frac{1}{2}} M_0^{\frac{1}{2}} x^{-\frac{1}{2}},$$

where ϵ is an experimental constant, which is related to the entrainment coefficient α and to the half-width growth rate $db(x)/dx$ through the relation

$$\epsilon = \alpha\sqrt{2} = \frac{1}{4}\left(\frac{2\pi}{\ln 2}\right)^{\frac{1}{2}} \frac{db}{dx}.$$

The jet growth rate db/dx varies from 0.09 to 0.12 and therefore ϵ varies from 0.068 to 0.09, but for the purpose of this paper an average value of the jet growth rate and of ϵ will be assumed (see List 1982*a*,*b*).

The initial conditions (nozzle shape, jet turbulence level) are predominant factors in the growth of the jet mixing layers for the first few jet diameters (e.g. up to $x \approx 15D$). However, we should point out that Flora & Goldschmidt (1969), performed an experimental investigation of the plane jet using nine different nozzle shapes (abrupt, gradual, circular, etc.) in an attempt to correlate the nozzle shape with variations in the jet growth rate. They reported that changes in the contracting section did not influence the spreading rate of the jet.

This agrees with the experimental results of Hussain & Clark (1977). The available experimental evidence is that the spreading rate of the plane jet a few diameters from the exit (e.g. for $x/D > 15$) is almost independent of the initial boundary conditions (see also Gutmark & Wygnanski 1976, for a similar comment). Kotsovinos (1975, 1976) presented convincing evidence that the reported jet spreading rates from various investigators exhibit substantial scatter because the growth of the jet is not linear on a large scale.

Therefore (7) can be integrated from $x = x_0$ to x and we obtain

$$M_1(x) = M_0 - \lambda M_0 \ln\frac{x}{x_0}, \tag{9}$$

where

$$\lambda = \epsilon\left[\frac{k(1+\tan^2\phi)}{4(-1+k\tan\phi)^2} - \frac{\tan\phi}{2(-1+k\tan\phi)}\right], \tag{10}$$

and where $M_1(x)$ is the first-order approximation to the jet momentum variation (decrease or increase). Equation (9) is identical with the solution proposed by Kotsovinos (1978*b*).

We will now try to find higher-order approximations for the jet momentum flux, as follows. The second-order approximation $M_2(x)$ of the variation of the jet momentum flux with x is found assuming that the (local) induced volume flux $\mathrm{d}V/\mathrm{d}x$ depends on the local momentum flux $M_1(x)$ given by (9) (and not on the initial momentum flux M_0), i.e. we take into account the reactive effect which the modification of the momentum flux exerts on the entrainment of ambient fluid,

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \epsilon^{\frac{1}{2}} M_1(x)^{\frac{1}{2}} x^{-\frac{1}{2}}. \tag{11}$$

A similar hypothesis has been adopted by Kotsovinos (1975, p. 37) and Schneider (1985). Combining (7), (9) and (11) we find

$$M_2(x) = \left[1-\lambda\ln\frac{x}{x_0}+\frac{\lambda^2}{2}\left(\ln\frac{x}{x_0}\right)^2\right]M_0. \tag{12}$$

In order to find the third-order approximation $M_3(x)$ for the jet momentum flux we now assume that the local entrainment $\mathrm{d}V/\mathrm{d}x$ depends on the local jet momentum flux $M_2(x)$ given by (12), i.e.

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \epsilon^{\frac{1}{2}} M_2(x)^{\frac{1}{2}} x^{-\frac{1}{2}}, \tag{13}$$

so that we obtain the third-order approximation for the jet momentum flux,

$$M_3(x) = \left[1-\lambda\ln\frac{x}{x_0}+\frac{\lambda^2}{2}\left(\ln\frac{x}{x_0}\right)^2-\frac{\lambda^3}{3!}\left(\ln\frac{x}{x_0}\right)^3\right]M_0. \tag{14}$$

In general, the n-order approximation $M_n(x)$ of the jet momentum flux is given by

$$\frac{M_n(x)}{M_0} = 1-\lambda\ln\frac{x}{x_0}+\frac{\lambda^2}{2}\left(\ln\frac{x}{x_0}\right)^2-\frac{\lambda^3}{3!}\left(\ln\frac{x}{x_0}\right)^3+(-1)^n\frac{\lambda^n}{n!}\left(\ln\frac{x}{x_0}\right)^n. \tag{15}$$

For $n\rightarrow\infty$ this series is equivalent to $(x/x_0)^{-\lambda}$, i.e.

$$\frac{M(x)}{M_0} = \left(\frac{x_0}{x}\right)^{\lambda}. \tag{16}$$

It is interesting to notice that the jet momentum flux $M(x)$ decreases when $\lambda > 0$, increases when $\lambda < 0$ and remains constant only when $\lambda = 0$. This may explain the experimental results of various investigators who found a decrease or an increase of momentum flux. The critical experimental parameter which determines the parameter λ is the angle ϕ.

The variation of the exponent λ with the angle ϕ is plotted in figure 2. It is apparent that the exponent λ is positive when $-7.08° < \phi$ and negative when $\phi < -7.08°$, i.e. when the induced flow is in the same direction with the jet flow and makes an angle ϕ less than $-7.08°$.

The dimensionless jet momentum flux $M(x)/M_0$ is calculated from (16) and is plotted in figure 3 for various values of the angle ϕ. It is easy to observe that the jet momentum is conserved only when $\phi = -7.08°$. For $\phi > -7.08°$ the jet momentum flux decreases and for $\phi < -7.08°$ it increases with the axial coordinate x. This is interesting because it gives a hint for an explanation of the increase of the jet momentum flux found by some experimentalists (see for example Hussain & Clark 1977). This point is further discussed below.

The exponent λ given by (10) can be written as

$$\lambda = \lambda_{\mathrm{p}}+\lambda_v,$$

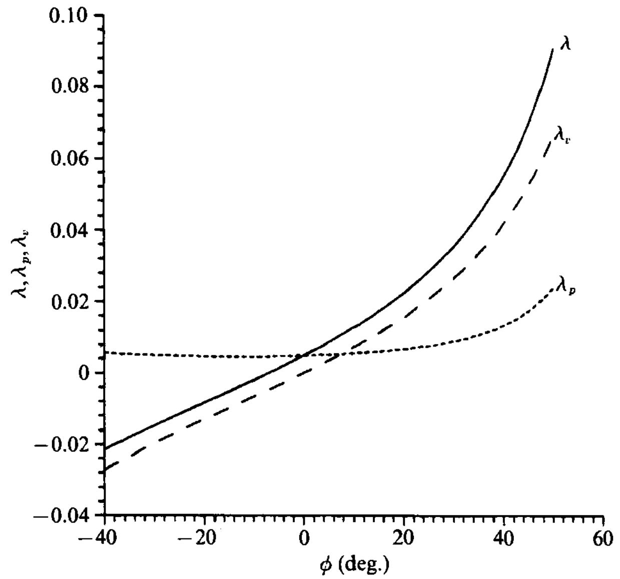

FIGURE 2. The parameters λ, λ_p and λ_v as a function of the induced flow angle ϕ.

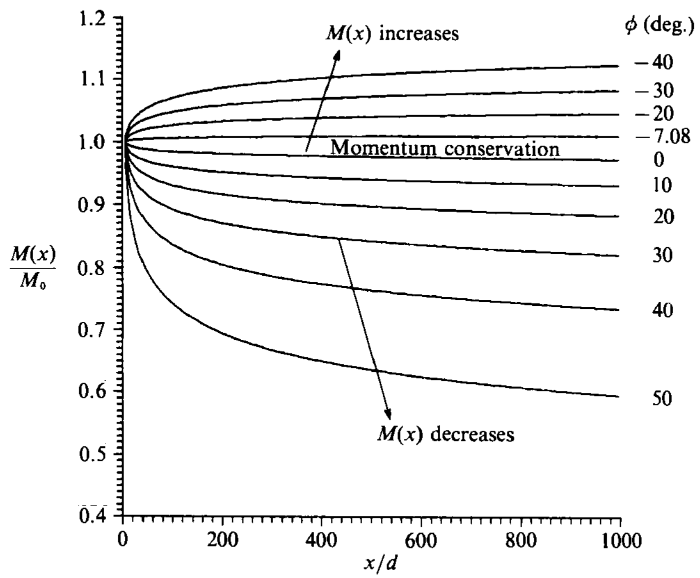

FIGURE 3. The theoretical prediction of the jet momentum flux $M(x)/M_0$ as a function of the axial distance x/D and the induced flow angle ϕ.

where
$$\lambda_p = \epsilon \frac{k(1+\tan^2\phi)}{4(-1+\tan\phi)^2} \tag{17a}$$
is the contribution of the induced pressure field and where
$$\lambda_v = \frac{\epsilon \tan\phi}{2(1-k\tan\phi)} \tag{17b}$$
is the contribution of the induced velocity field (see figure 2).

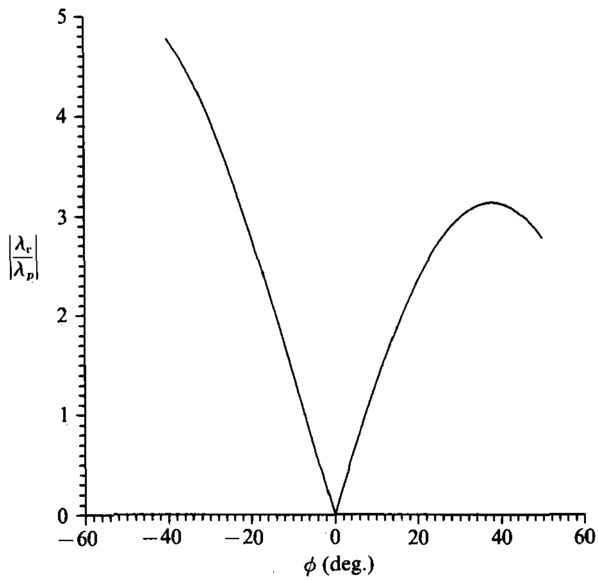

FIGURE 4. The ratio $|\lambda_v/\lambda_p|$ (see (17a, b)) as a function of the angle ϕ.

The ratio $|\lambda_v/\lambda_p|$ is a measure of the importance of the induced velocity field relative to the pressure field in decreasing the jet momentum flux. This ratio is plotted in figure 4 as a function of the angle ϕ. It is apparent from figure 4 that for $-40° < \phi < 40°$ the ratio $|\lambda_v/\lambda_p|$ varies from zero to 4.8, which implies that the contribution of the pressure field is important and should not be neglected, e.g. for $\phi = 2°$ the contribution of the pressure field in decreasing the jet momentum flux is several times larger than the contribution of the induced velocity field. The relative contribution of those two factors is also indicated in figure 5(a) where $M(x)$, $C(x)$ (the contribution of the momentum flux of the induced flow) and $H(x)$ (the contribution of the induced pressure) are calculated and plotted as a function of the axial distance x/D for an induced flow angle $\phi = 45°$ (a plane jet out of a wall). It is clear that the contribution of the induced pressure in decreasing the momentum flux is significant.

The terms $H(x)$ and $C(x)$ at an axial distance $x/D = 150$ are plotted as a function of the angle ϕ in figure 5(b). We observe that the pressure term $H(x)$ is always negative and varies slightly with the angle ϕ. In contrast, the induced flow term $C(x)$ varies considerably with the angle ϕ, taking both positive and negative values. For $\phi < 0°$, the induced flow adds momentum flux to the jet momentum flux. For $\phi = -7.08°$, $C(x)+H(x) = 0$, i.e. the (positive) induced flow momentum flux $C(x)$ cancels the negative contribution of the induced pressure force $H(x)$ and the jet momentum flux is conserved, i.e. $M(x) = M_0$.

The theoretical solution presented in this paper, given by (16), is compared below with the solutions given by previous investigators. Kraemer (1971) found that for a plane jet out of a wall the momentum flux of the induced flow field in a plane perpendicular to the jet axis from $y = -\infty$ to $y = +\infty$ is $0.03M_0$. His calculations are not relevant to the problem we study (see Kotsovinos 1978b for discussion). Kotsovinos (1975, p. 37) argued that the jet momentum flux for a plane jet out of a wall is not constant but decreases (in agreement with reliable experimental results) owing to the induced pressure and to the induced flow momentum flux along the jet

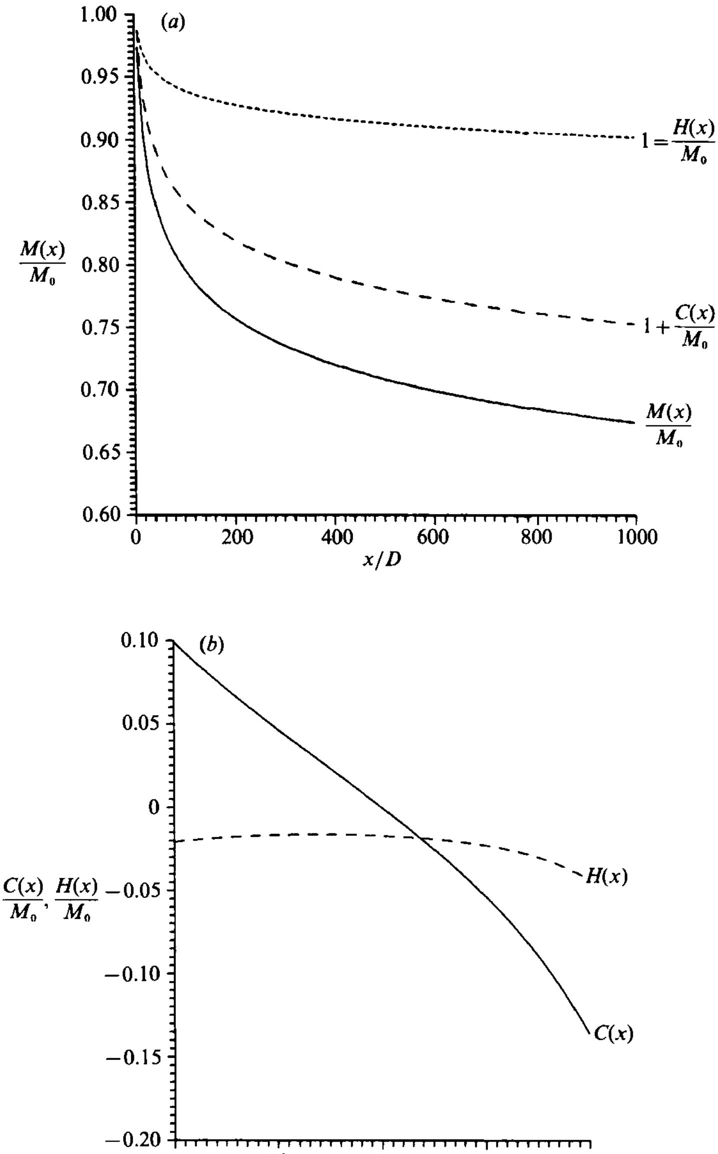

FIGURE 5. (a) The contribution of the induced flow momentum flux $C(x)$ and of the pressure term $H(x)$ in decreasing the jet momentum flux $M(x)$, for $\phi = 45°$ (a plane jet out of a wall). $M(x) = M_0 + C(x) + H(x)$. (b) The induced flow momentum flux $C(x)$ and the pressure term $H(x)$ (see (5) and (6)) calculated at $x/D = 150$ as a function of the angle ϕ.

boundaries. By neglecting the pressure field and assuming that the volume flux is based on the local momentum flux $M(x)$ he found that

$$\frac{M(x)}{M_0} = \left(\frac{x_0}{x + x_0}\right)^{\epsilon \tan\phi / 2(1 - k \tan\phi)}, \tag{18}$$

where the constant x_0 was put equal to $3.3D$. Apparently the exponent in (18) is equal to λ_v (see (17b)).

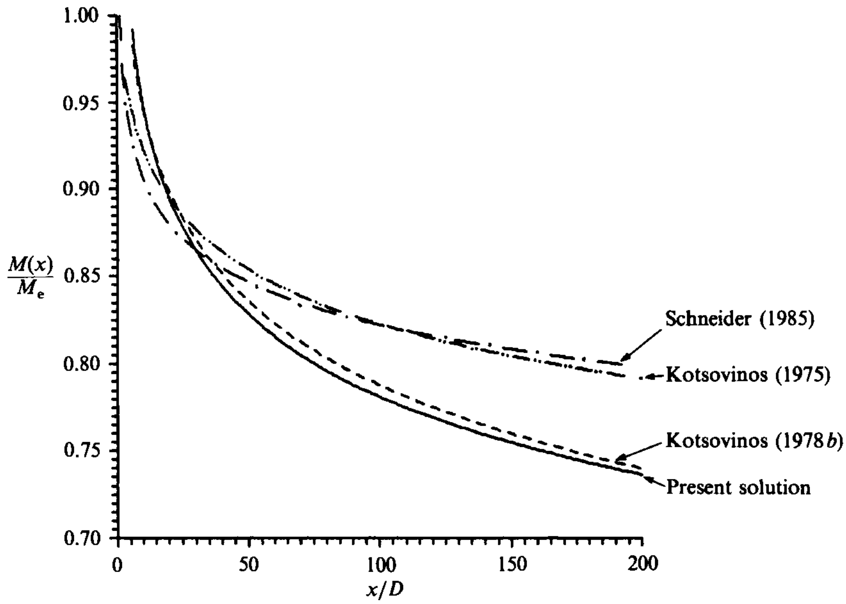

FIGURE 6. Theoretical models for the decay of the momentum flux of a two-dimensional jet out of a wall ($\Theta_w = 90°$).

Later, Kotsovinos (1978*b*) included the induced pressure field in his solution but he neglected the effect which the decrease of the momentum flux exerts on the induced flow, stating that his solution (see (1)) is a valid approximation for values of x/D of the order of a few thousands.

Schneider (1985) also neglected the pressure field and the jet expansion (i.e. he assumed $k = 0$) and found, using inner and outer expansions, that

$$\frac{M(x)}{M_0} = \left(\frac{x_0}{x}\right)^{\epsilon \tan\phi/2}.$$

Apparently, for $k = 0$, his solution is identical with Kotsovinos's (1975) solution, given by (18). The constant x_0 could not be determined and Schneider made the assumption that $x_0 = D$. However, x_0 represents (physically) a length of flow development region approximately equal to $4D$ to $6D$.

These theoretical models are compared in figure 6 with the solution of this paper (equation (16)). Kotsovinos's (1975) and Schneider's (1985) solutions appear to underestimate the decay of the jet momentum flux because they neglected the contribution of the induced pressure.

3. The modified induced flow field

The analysis of the previous section indicated the importance of the angle ϕ at which the induced flow streamlines enter the jet. The flow induced by a jet has been studied by Stewart (1956), Taylor (1958), Wygnanski (1964), Kraemer (1971) and Schneider (1981). Taylor (1958) assumed that the flow induced by a jet is to a first approximation an inviscid potential flow. He also assumed that, since the jet kinematic mass flux $V(x)$ increases with x, the ambient fluid sees the jet as a distribution of sinks along the jet axis of strength $\mathrm{d}V/\mathrm{d}x$ based on constant momentum flux, given by the relation

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \epsilon^{\frac{1}{2}} M_0^{\frac{1}{2}} x^{-\frac{1}{2}}.$$

Assuming the geometry of figure 1, he found that the induced flow streamlines enter the jet axis at a constant angle $\phi = \phi_{\mathrm{T}}$ which is given by Taylor (1958) or Kraemer (1971)

$$\phi_{\mathrm{T}} = 90° - \tfrac{1}{2}\Theta_{\mathrm{w}}. \tag{19}$$

Below, we explore the modification of Taylor's solution due to the variation of the jet momentum flux. It is apparent that there is an inter-relationship between the jet momentum flux and the induced flow, i.e. the induced flow field modifies the jet momentum flux which in turn modifies the jet entrainment and the induced flow field. It seems therefore reasonable to assume that the strength of the sinks along the jet axis depends on the local momentum flux $M(x)$, i.e.

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \epsilon^{\frac{1}{2}} M(x)^{\frac{1}{2}} x^{-\frac{1}{2}}$$

or, in combination with (16),

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \epsilon^{\frac{1}{2}} M_0^{\frac{1}{2}} \left(\frac{x_0}{x}\right)^{\lambda/2} x^{-\frac{1}{2}}.$$

Assuming slip conditions at the solid walls which bound the induced flow (see figure 1) and infinite space filled with ambient fluid without stratification, the induced flow is irrotational because at infinity the vorticity is zero. Schneider (1981) pointed out that the error resulting from assuming slip conditions at the solid walls is negligible for plane jet flow (but non-negligible for a round jet).

The induced flow streamlines $\Psi(x,y)$ satisfy:

(i) the Laplace equation

$$\Psi^2(x,y) = 0; \tag{20}$$

(ii) the slip condition at the solid walls $\Psi(x,y) = 0$;

(iii) the boundary condition along the jet axis (see Kotsovinos 1978*a*)

$$2\frac{\mathrm{d}}{\mathrm{d}x}(\Psi(x,0)) = \frac{\mathrm{d}V}{\mathrm{d}x} = \epsilon^{\frac{1}{2}} M_0^{\frac{1}{2}} \left(\frac{x_0}{x}\right)^{\lambda/2} x^{-\frac{1}{2}}. \tag{21}$$

The exponent λ (see (10)) depends on the angle ϕ which, in turn, is determined from the streamline $\Psi(x,y)$ (i.e. from the solution of the above problem). However, for a given value of the coefficient λ, we find that the stream function that satisfies the Laplace equation and the boundary conditions (20) and (21) is given in cylindrical coordinates by the similarity solution

$$\Psi(r,\theta) = \Gamma r^{(1-\lambda)/2}[\cos\tfrac{1}{2}(\theta(1-\lambda)) - A\sin\tfrac{1}{2}(\theta(1-\lambda))] \tag{22}$$

where

$$\Gamma = \frac{1}{1-\lambda}\epsilon^{\frac{1}{2}} M_0^{\frac{1}{2}} \left(\frac{1}{x_0}\right)^{-\lambda/2} \tag{23}$$

and

$$A = \frac{1}{\tan\frac{1}{2}\Theta_{\mathrm{w}}(1-\lambda)}. \tag{24}$$

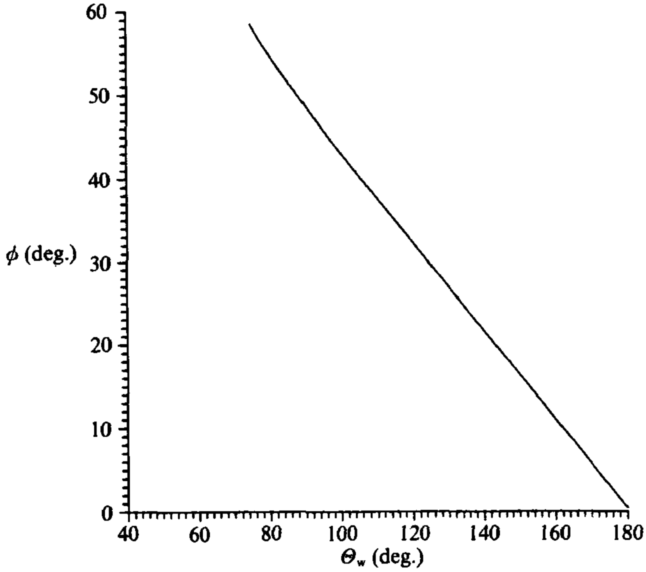

FIGURE 7. The modified solution for the induced flow angle ϕ as a function of the angle Θ_w between the bounding walls.

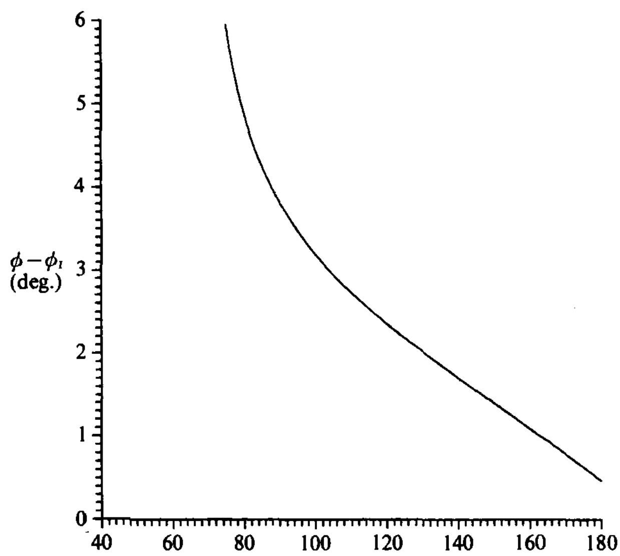

FIGURE 8. The difference between the modified induced flow angle ϕ (equation (25)) and the angle ϕ_T (Taylor's solution, (19)) plotted as a function of the angle Θ_w between the bounding walls.

The modified induced flow streamlines $\Psi(r, \theta)$ (given by (22)), enter the jet axis at an angle ϕ that is implicitly given by the equation

$$\phi = 90^\circ - \tfrac{1}{2}\Theta_w(1-\lambda). \tag{25}$$

Using a computer and by the method of iterations we can find ϕ as precisely as we want: starting from Taylor's solution (equation (19)) we determine the coefficient λ from (10), and then calculate A from (24) and ϕ from (25). With the new value of ϕ, we obtain a better approximation for the coefficient λ, for A and for ϕ and so on. The

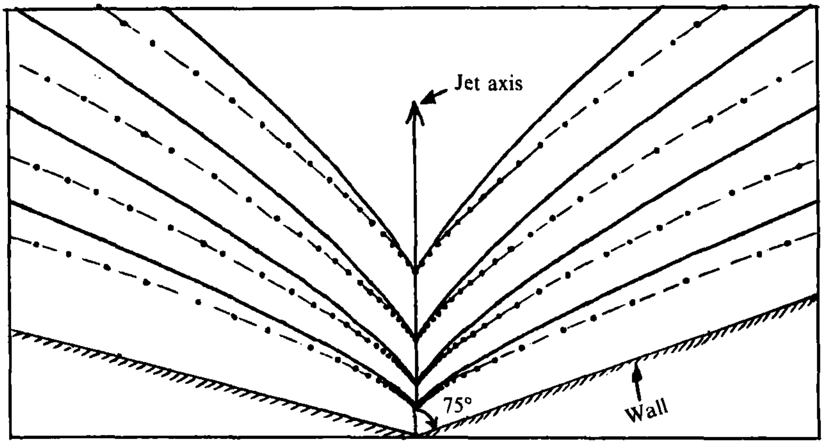

FIGURE 9. Streamlines of the flow induced by a plane jet emerging from a corner of angle $\Theta_w = 75°$. ——, Present solution; —·—·—·—, solution with constant momentum flux (Taylor 1958).

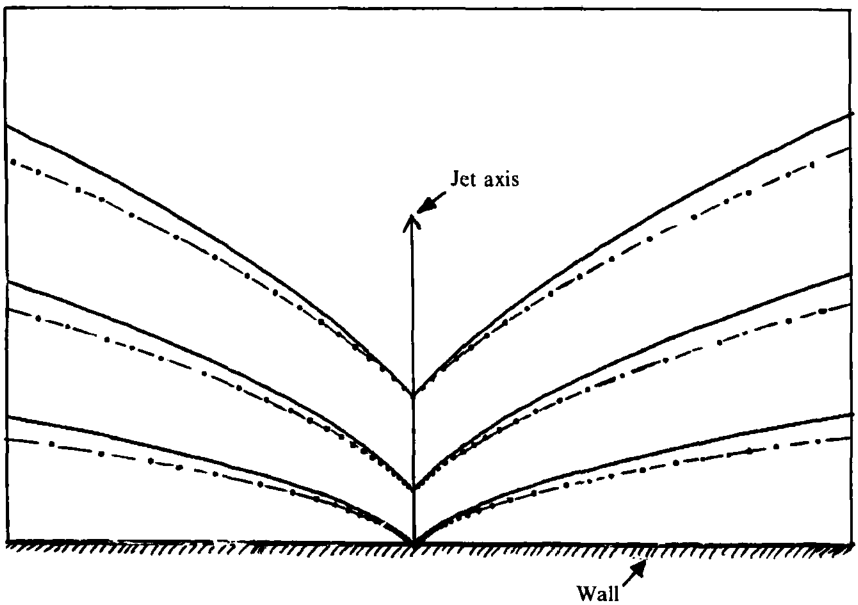

FIGURE 10. Streamlines of the flow induced by a plane jet emerging from a plane wall ($\Theta_w = 90°$). ——, Present solution (non-constant momentum flux); —·—·—·—, solution with constant momentum flux (Taylor 1958).

iterations stop when the difference between the new value for ϕ and its previous value is less than 0.001°.

The modified solution for the angle ϕ is plotted in figure 7 as a function of the angle Θ_w to the bounding walls. For a plane jet out of a wall ($\Theta_w = 90°$) we predict $\phi = 48.7°$. The difference between the angle ϕ calculated in this paper (equation (25)) and the angle ϕ_T from Taylor (1958) is equal to $\frac{1}{2}\Theta_w \lambda$ and is plotted in figure 8 as a function of the angle Θ_w. There appears to be a difference between these two solutions.

The modified induced flow streamlines given by (22) are drawn in figures 9, 10 and 11 for Θ_w equal to 75°, 90° and 180° respectively, together with the streamlines from Taylor (1958) for comparison; it is observed that our solution predicts that the

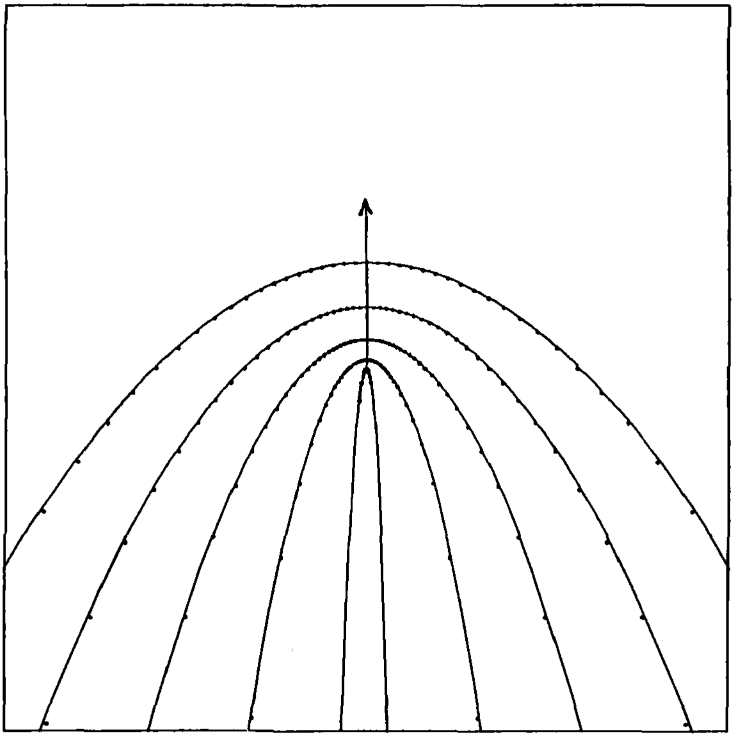

FIGURE 11. Flow streamlines induced by a plane jet in open space, $\Theta_w = 180°$. ——, Taylor's solution (1958); —·—·—·—, present solution.

streamlines enter the jet at an angle which is steeper than the classical Taylor's solution for $\Theta_w \leqslant 90°$, but that for $\Theta_w = 180°$ (jet in open space) the streamlines of both solutions practically coincide.

Van Dyke (1982, picture 169) shows the flow induced by a plane jet in open space, where it is observed that the induced flow streamlines are parabolas, entering the jet axis at constant angle ϕ. Kotsovinos (1975) and Goldschmidt, Moallemi & Oler (1983) observed a negative longitudinal component of the mean velocity vector at the edges of a plane jet out of a wall, in agreement with the theoretical prediction. Giger (1987, p. 46) and Lippisch (1958) observed the induced flow streamlines in a plane jet out of a wall. Their experimental results indicate a small difference between the experimentally determined streamlines and Taylor's streamlines, in qualitative agreement with the modified streamlines plotted in figure 10 (see also Kraemer 1971).

4. Comparison with experimental results

In order to compare the theoretical solution with the experimental findings, it is necessary to take into account the following remarks:

(i) Experimental results which correspond very closely to the flow configuration of figure 1 can be strictly compared with the theoretical solution for the variation of the jet momentum flux (16). The bounding walls of figure 1 should extend to a transverse distance many times the axial distance where measurements are taken. The jet aspect ratio should be small to avoid secondary flow effects (see Foss & Jones 1964 or Giger 1987).

(ii) The calculated induced flow field is based on an idealization of the flow geometry (i.e. infinite space, infinite bounding walls, infinite jet axis) and on an idealization of the ambient fluid (i.e. homogenous fluid at exactly the same temperature everywhere, with no ambient currents except for the currents induced by the jet itself). Owing to the finite dimensions of the laboratory rooms (for air jets) or of the water tanks (for the water jets) a large recirculation is induced in practice which influences the induced flow streamlines. Most important, weak density (or temperature) stratification of the ambient fluid may seriously modify the angle ϕ. In that case knowledge of the direction of the jet flow relative to the vertical is essential. The theoretical solution for the induced flow angle ignores the fact that for $x < x_0$ the flow resembles the flow of a mixing layer and is very sensitive to initial conditions. Therefore, given enough space, the angle ϕ becomes constant at distances $x > x_1$, where x is of the order of x_0. For $x = 0$ the induced flow is almost parallel to the bounding wall and therefore the angle $\phi(0)$ is $90° - \Theta_w$. It is assumed that in the region $0 < x < x_1$ the angle $\phi(x)$ varies linearly from $\phi(0)$ to $\phi(x_1)$.

(iii) The theoretical solution concerns the variation of the jet momentum flux $M(x)$, as defined by (4), i.e. $M(x)$ includes the contribution from longitudinal turbulence and mean static pressure, but most experimentalists measured only the axial mean velocity profile. However, the experimental results of Miller & Comings (1957), Bradbury (1965) and Hussain & Clark (1977) indicate that approximately

$$\int_{-B(x)}^{B(x)} \left(\overline{u}'^2 + \frac{P}{\rho}\right) \frac{\mathrm{d}y}{M(x)} = 0.$$

Therefore, the momentum flux $M(x)$ can be estimated from mean axial velocity measurements assuming that

$$M(x) = \int_{-B(x)}^{B(x)} \overline{u}^2(x, y)\, \mathrm{d}y.$$

(iv) For $y < 0.17|x|$, it is sufficient to measure accurately the axial velocity profile $u(x, y)$ in the jet flow in order to have a very good estimation of the momentum flux $M(x)$ (see Kotsovinos 1978*b*). This is an important point because existing experimental results are inaccurate close to the jet boundaries (i.e. for $y > 0.17|x|$ where flow reversals are observed, see Kotsovinos 1975, 1977, and Goldschmidt *et al.* 1983). These measurements require instrumentation capable of distinguishing flow direction. However, we point out again that these inaccuracies do not greatly influence the magnitude of the experimentally calculated momentum flux.

Below, we compare the experimental results for a plane jet out of a wall with the theoretical prediction. We choose the plane jet out of a wall because it has been extensively investigated and good reliable data are available. The experimental results of Heskestad (1965), Kotsovinos (1975), Miller (1957), Giger (1987), Goldschmidt & Eskinazi (1966) are compared in figure 12(*a–e*) with the theoretical prediction of this paper, (16). The constant angle ϕ which fits the experimental data of Heskestad is identical with the theoretical angle 48.7° and for the other experiments varies from 43° to 59.5°. It was also assumed that the induced flow angle ϕ becomes constant for $x > x_1$, and that in the region $0 < x < x_1$, ϕ increases linearly with x, from 0° to ϕ. This distance x_1 was equal to $x_0 = 6D$ for all the experiments, with the exception of Goldschmidt & Eskinazi's (1966) experimental results, where $x_1 = 3x_0$. The constants ϵ and C_1 were equal to 0.0784 and 0.07 respectively.

All the experimental results in figure 12 appear to follow the trend predicted by the

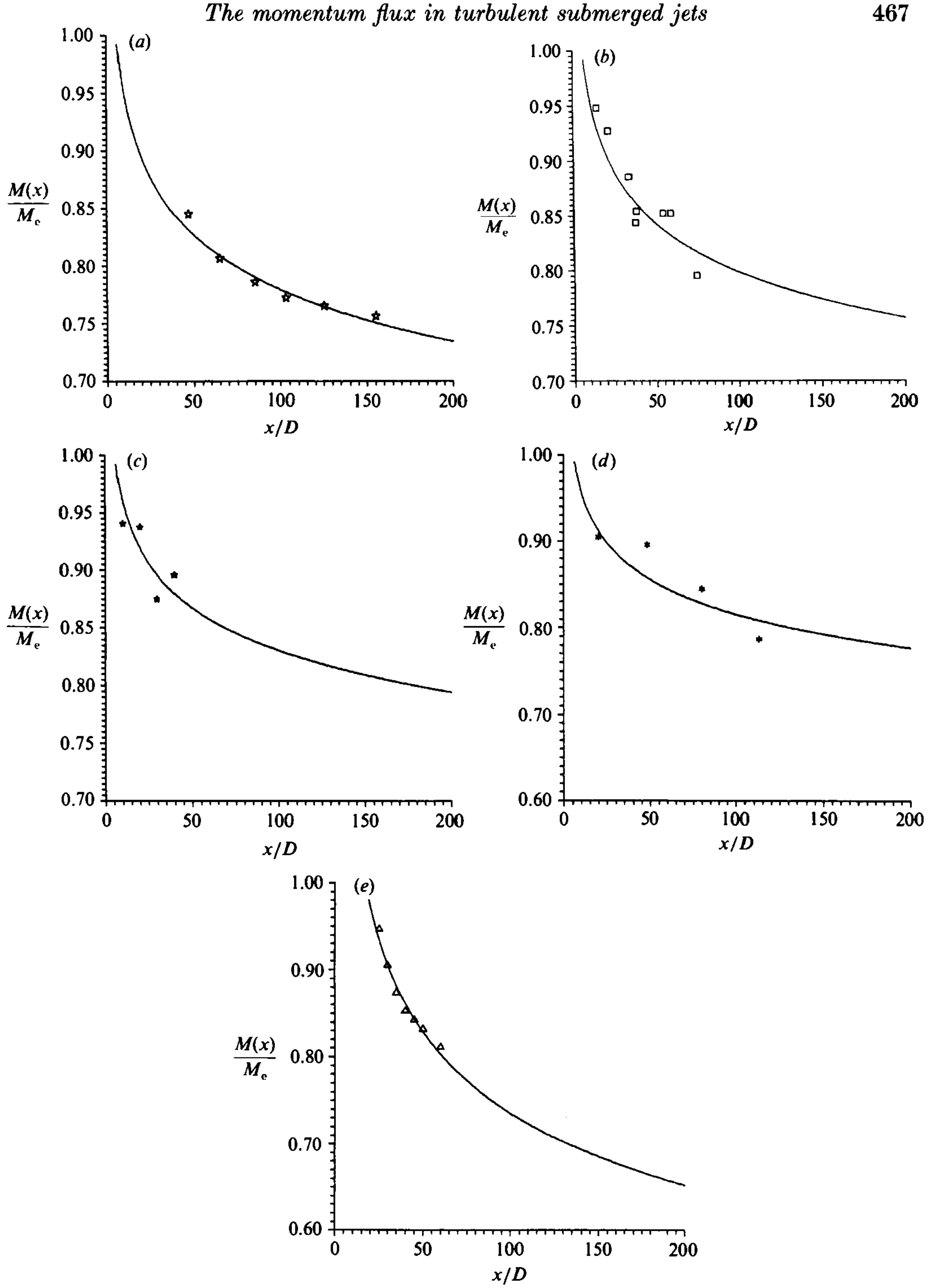

FIGURE 12. Comparison between experimental data and theoretical predictions. (*a*) Data from Heskestad (1965) and theoretical prediction with $\phi = 48.7°$; (*b*) Kotsovinos (1975) and $\phi = 47°$;(*c*) Miller (1957) and $\phi = 43°$; (*d*) Giger (1987) and $\phi = 45°$; (*e*) Goldschmidt (1964) and $\phi = 59.5°$.

theory. However all these experimental data from various investigators do not coincide. Assuming that all these results are reliable, we may argue that this is due (i) to differences of the induced flow in the tank (or room), (ii) to differences in the initial mixing layers.

5. Concluding remarks

The balance (conservation) of momentum flux for the fluid contained within a closed surface defined by the jet boundaries $y = \pm kx$ and the jet cross-section at distance x disclosed that the initial (input) jet momentum flux is equal to the sum of the jet momentum flux $M(x)$ (which includes contributions from longitudinal turbulence and mean static pressure) and of the fluxes along the jet boundaries of the induced flow momentum and of the pressure force.

The theoretical analysis assumed steady-state jet flow. Since the jet velocity is finite, steady-state induced flow and pressure fields are achieved at infinite time. The theoretical solution presented in this paper appears to indicate that for $\lambda < 0$ and $x \rightarrow \infty$ the jet momentum flux $M(x) \rightarrow \infty$, that for $\lambda = 0, M(x) = M_0$, and that for $\lambda > 0$ and $x \rightarrow \infty$, $M(x) \rightarrow 0$.

The angle ϕ at which the induced flow streamlines enter the jet is found theoretically assuming infinite space, the jet extending to infinity and a steady-state solution. The jet entrainment is simulated by a distribution of sinks along the jet axis of strength corresponding either to constant (Taylor solution) or variable (this paper) momentum flux. The minimum value of the theoretically calculated angle ϕ for the geometry of figure 1 is obtained for $\Theta_w = 180°$ (jet in open space), and is zero according to Taylor's solution (see (19)) and equal to $-0.5°$ according to this paper. Moreover the theoretical solution predicts that the angle ϕ is independent of the x-axis. The explicit solution for the variation of $M(x)$ with distance x (equation (16)) was made possible because we considered that the angle ϕ is independent of the x axis, from $x = x_0$ to infinity.

The theoretical analysis presented in this paper indicates that the jet momentum flux $M(x)$ decreases for $\phi > -7.08°$, remains constant for $\phi = -7.08°$ and increases for $\phi < -7.08°$. Since the minimum value of the theoretically calculated angle ϕ is $-0.5°$ (plane jet in open space, $\Theta_w = 180°$), we could argue that reliable experimental data that indicate constancy or increase of the momentum flux $M(x)$ are associated with angles ϕ much smaller than $-0.5°$, i.e. with angles ϕ that deviate considerably from the theoretical (irrotational, inviscid) induced flow field. It seems reasonable to assume that, owing to finite dimensions of laboratory rooms, or to viscous effects along the bounding walls and weak temperature stratification, the angle ϕ may attain values much smaller than zero and, in addition, may depend on the axial distance x. Moreover, the experimental results of Bradshaw (1966), Bradshaw (1977), Goldschmidt & Bradshaw (1981), Hussain & Clark (1977), indicate that up to $x/D = 15$, there is a systematic dependence of the jet widening, mean centreline velocity decay and volume (or mass) flux, on mean and turbulent characteristics of the initial (upstream) boundary layer. Since the volume flux is related to local momentum flux, i.e. to the strength of sinks and to the induced flow, it is clear from the analysis of this paper that the initial conditions influence the angle ϕ at which the induced flow streamlines enter the jet.

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